Ch. 3: The Philosophy of Mathematics

① What ~~is true?~~ is meaningful?

Motiv.? Gödel's Incompleteness Theorems

* "Not everything can be proven"

* Colloquially, this statement roughly translates to the non-provability of a statement like "This statement is false."

② Three main philosophies

Platonism ⟷ Religion | Infinite sets make sense "in my ideal universe"

Formalism ⟷ Agnostic "Whatever is convenient"

Finitism ⟷ Athiest Infinite sets are nonsense

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$$(1 + x + x^2 + \cdots)(1 - x)$$

$$= 1 + \cancel{x} - \cancel{x} + \cancel{x^2} - \cancel{x^2} = 1$$

Case: $(\beth_\omega)^{\aleph_0} > \beth_\omega$ (true by ZFC)

* There are many infinite objects here.

* ~~Platonists~~ Finitists might agree w/ this informally but would think it's meaningless.

$1, \omega, \aleph_0, \aleph_1, P(\mathbb{R}) \qquad +\infty$

* The finitists may further be said to be "intuitionist" or "constructionist".

* With regard to the latter, $\varphi \vee \neg\varphi$ is sometimes meaningless.

(e.g., let φ = twin prime conjecture). $\varphi \vee \psi$

Rmk: Gödel's Incompleteness Theorems can be formulated for the finitist point of view.

Assumption: You are at least a formalist who has decided to stay. If not...

* We must then be honest about 3 things

1) Formal logic is developed twice: $\emptyset \longrightarrow \text{Meta} \longrightarrow \text{ZFC}$
2) We assume ZFC is consistent by contradiction ($\text{ZFC} \nvdash \text{Con}(\text{ZFC})$).
3) Theorems are <u>in principle</u> formal theorems of ZFC. (because no one will translate Rudin to such a thing)

③ The Axiom of Choice

you forgot something?

* ZFC = ZF + AC. Let us play w/ this by considering the following:

?: Many ZF proofs say "choose x s.t. $\varphi(x)$ . . .". More formally,

$$\exists x\, \varphi(x),\ \ \forall x\,(\varphi(x) \Rightarrow \psi) \vdash \psi$$

This resolves to

$$\forall x\,(\varphi(x) \longrightarrow \psi),\ \neg\psi \vdash \forall x\, \neg\varphi(x)$$

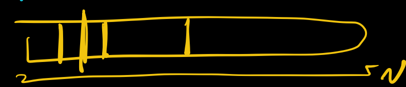
where any notion of choice vanishes.

* This will validate the "EI rule" (Lemma 2.11.8, Kunen 2012).

Go on...

Ch. 4: Recursion Theory

Defn: $\mathcal{C} \subseteq \omega^{\omega}$ is the set of computable functions (e.g., in the sense of Turing)



$\log N$

$\lfloor \log_2 N \rfloor + 1$

* For us, computability of a function $f: D^n \to D$ is one that lands in D after finite time.

Defn: A relation $R \subseteq D^n$ is decidable if $\exists$ a "computable function" $D^n \to \{T, F\}$.

$(x, y) \in R \iff x \sim y$

Defn: Δ_0 will be a set of trivially decidable relations, depending on D. "$D = HF$"

Defn: Δ_1 is the subset of Δ_0 subject to decidability.

Church-Turing: ($D = HF$) $R \subseteq HF^n$ decidable $\iff$ R is Δ_1. A function $f: HF^n \to HF$ is computable when f is a Δ_1 subset of HF^{n+1}. $\{(x, f(x)) : x \in HF^n\} \subseteq HF^{n+1}$

Rmk: HF^{n+1} says to interpret $f \subseteq HF^{n+1}$ as the graph of f.

Rmk: HF is "hereditarily finite set" $\longleftrightarrow$ each set consists of sets which are HF, and the terminating condition is $\emptyset$.

$\{6, 7\}$ $\{\emptyset, \{\emptyset\}\}$ $\{\mathbb{Z}\}$ ✗ HF

✓ HF